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Beam-envelope theory of ionization cooling[☆]

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Abstract

Linear beam-envelope theory of ionization cooling in 6D phase space has been systematically established in the past few years. In this paper, we briefly review the general formalism as well as the specific theories for a quadrupole channel and a bent-solenoidal channel with symmetric focusing. These channels play important roles in the design of cooling channels for the envisioned neutrino factories and muon colliders. The analytical solutions of these channels are relatively simple yet provide good understanding of cooling and heating mechanisms in both transverse and longitudinal phase spaces. Furthermore, the resulting formulae can be used to evaluate cooling channel designs the same way as the radiation integrals are used in storage ring designs.

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1. Introduction

Ionization cooling is the only promising process that can substantially cool a muon beam within a muon's short lifetime and make it useful for the envisioned neutrino factories and muon colliders [1–5]. The mechanism of ionization cooling is similar to the mechanism of radiation damping

despite the differences in fundamental interactions. In both cases, particles lose incoherent momentum by either radiating photons or ionizing atoms, and gain coherent momentum through an accelerating field only in the beam forward direction. Coexisting with this cooling mechanism are heating mechanisms due to stochastic processes: quantum excitation in case of radiation damping, and multiple scattering and energy straggling in case of ionization cooling. The similarity suggests that the well-established formalism for radiation damping [6–12] can be adapted for ionization cooling. Neuffer [13] has treated ionization cooling intuitively as Sands [6] did for radiation damping. In recent years, ionization cooling has been treated with the beam moment equation approach by

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many authors [14–20] and a linear beam-envelope theory has been established, which is an analog of the theory of radiation damping [11]. This theory provides a better understanding of ionization cooling, unveils new heating mechanisms, and can be used to estimate cooling channel performance. In the following we briefly review this theory.

2. General formalism of the beam-envelope theory

In our discussion we work with the phase-space vector $X = (x, p_x, y, p_y, z, \delta)^T$, where (x, y, z) are positions relative to the reference particle, p_x and p_y are transverse canonical momenta normalized by the reference momentum p_0 , and $\delta = (E - E_0)/\beta_0^2 E_0 \approx \Delta p/p_0$.¹ The linearized equation of motion using path length s as the time variable is of the general form

$$\frac{dX}{ds} = JHX + AX + \Xi. \quad (1)$$

Here, the first term on the right-hand side is the Hamiltonian part of the motion, where J is the symplectic identity matrix,² and H is the symmetric matrix associated with the Hamiltonian \mathcal{H} via $\mathcal{H} = X^T H X / 2$. The Hamiltonian part contains the dominating macroscopic forces that provide focusing, acceleration, and dispersions if necessary. The last two terms represent the microscopic interactions with materials giving rise to ionization damping (the second term) and stochastic excitations (the third term) from multiple scattering and energy straggling.

For the Hamiltonian part, we consider a general cooling channel that uses either a solenoidal field or a quadrupole field or both for focusing, and a vertical dipole field for dispersion. The magnetic field in the usual Frenet–Serret coordinate system $\{x, y, s\}$ can be written, up to the linear

order, as

$$B_x(x, y, s) = -\frac{1}{2}b'_s x + b_1 y \quad (2a)$$

$$B_y(x, y, s) = b_0 + b_1 x - \frac{1}{2}b'_s y \quad (2b)$$

$$B_s(x, y, s) = b_s - \kappa b_s x + b'_0 y \quad (2c)$$

where b_s , b_0 , b_1 are the solenoidal, dipolar, quadrupolar components. A prime indicates differentiation with respect to s . $\kappa(s)$ is the curvature of the reference orbit and is normally chosen to be $\kappa(s) = qb_0(s)/p_0$ for a reference particle of charge q and nominal momentum p_0 .

The Hamiltonian governing particle motion in a linear approximation in the above magnetic field can be worked out as³ [21]

$$\begin{aligned} \mathcal{H} = & \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\hat{b}_s^2(x^2 + y^2) - \hat{b}_s L_z \\ & - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{1}{2}\hat{b}_1(x^2 - y^2) \\ & + \frac{1}{2}\left[\frac{1}{\gamma_0^2}\delta^2 + V(s)z^2\right] \end{aligned} \quad (3)$$

where

$$\hat{b}_s = \frac{q}{2p_0}b_s = \frac{q}{2p_0}B_s(0, 0, s)$$

$$\hat{b}_1 = \frac{q}{p_0}b_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x}\bigg|_{x=y=0}$$

$$\frac{1}{\rho} = \kappa = \frac{q}{p_0}b_0 = \frac{q}{p_0}B_y(0, 0, s)$$

$$L_z = xp_y - yp_x$$

is the canonical angular momentum, and γ_0 is the Lorentz factor of the reference particle. Note that only the zero-order longitudinal field b_s plays a role here because the Lorentz force resulting from the longitudinal field already depends on the small transverse momentum. Furthermore, the field components involving b'_s do not play an explicit role because we are working with canonical momentum. For the longitudinal motion we added a simple oscillator with focusing strength $V(s)$ and assumed no acceleration of the reference particle.

¹Note that, since the momenta are normalized by p_0 , the emittances in our discussion are the geometrical emittances. Our coordinate system is the same as the one used in the MAD program except for a scaling factor β_0 in the longitudinal variables.

² $J = \text{diag}(J_2, J_2, J_2)$ and $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

³The $1/\gamma_0^2$ factor was omitted in a couple of our early papers including Ref. [20]. Thanks to Neuffer for pointing this out.

This Hamiltonian covers most of the cooling channels under consideration.

For the material part of the equation of motion, the following simple model is used to describe the microscopic forces due to ionization and multiple scattering:

$$\left. \frac{dx}{ds} \right|_M = \left. \frac{dy}{ds} \right|_M = \left. \frac{dz}{ds} \right|_M = 0 \quad (4a)$$

$$\left. \frac{dp_x}{ds} \right|_M = -\eta(p_x + \hat{b}_s y) + \sqrt{\chi} \zeta_x^{\text{MS}} \quad (4b)$$

$$\left. \frac{dp_y}{ds} \right|_M = -\eta(p_y - \hat{b}_s x) + \sqrt{\chi} \zeta_y^{\text{MS}} \quad (4c)$$

$$\left. \frac{d\delta}{ds} \right|_M = -(\partial_\delta \eta) \delta - (\partial_x \eta) x - (\partial_y \eta) y + \sqrt{\chi_\delta} \zeta_z^{\text{ES}}. \quad (4d)$$

In this model, interaction is instantaneous, and thus there is no change in position. Since ionization affects the magnitude but not the direction of the particle momentum, the kinematic momentum change $d\vec{p}/ds = dp/ds \vec{p}/p = -\eta \vec{p}$ where

$$\eta \equiv \frac{1}{pv} \left| \frac{dE}{ds} \right| \quad (5)$$

is a positive quantity characterizing the average force due to ionization energy loss for a particle of kinematic momentum p and velocity v . This leads to the first r.h.s. terms in the transverse momentum equations, where the vector potential terms $\hat{b}_s x$ and $\hat{b}_s y$ convert canonical momentum to kinematic momentum. Since the effect of ionization energy loss on the reference particle is assumed to be compensated by acceleration, $d\delta/ds$ is affected only by deviations from the average loss, which leads to the first three terms in the energy equation.

Effects of multiple scattering and energy straggling are modelled as random kicks to the momentum and energy. Multiple scattering is characterized by the projected mean-square angular deviation per unit length

$$\chi = \left(\frac{13.6 \text{ MeV}}{pv} \right)^2 \frac{1}{L_{\text{rad}}} \quad (6)$$

where L_{rad} is the radiation length of absorbers. Energy straggling is characterized by the mean-

square relative energy deviation per unit length χ_δ . The stochastic nature of multiple scattering and energy straggling are modelled by the uncorrelated unit stochastic quantities $\zeta_x^{\text{MS}}(s)$, $\zeta_y^{\text{MS}}(s)$, and $\zeta_z^{\text{ES}}(s)$. To avoid complexity arising from the geometry of absorbers, we assume absorbers have uniform thickness but certain density variation to account for the thickness variation (in wedged absorbers, for example). Thus η , χ , and χ_δ may depend on a muon's transverse position through density variation. The properties of ionization, energy straggling, and multiple scattering had been extensively studied, and considered by Neuffer in the context of ionization cooling as well. In particular, the energy loss rate dE/ds is given by the Bethe–Bloch formula, and the χ_δ can be estimated by [5,22,23]

$$\begin{aligned} \chi_\delta &= \frac{1}{(pv)^2} \frac{d\langle \Delta E_{\text{rms}}^2 \rangle}{ds} \\ &\simeq 4\pi N_A r_e^2 \left(\frac{m_e}{m_\mu} \right)^2 \rho \frac{Z}{A} \frac{1}{\beta^4} \left(1 - \frac{\beta^2}{2} \right) \end{aligned} \quad (7)$$

where N_A is the Avogadro constant; r_e is the classical radius of an electron; m_e and m_μ are the masses of the electron and muon; ρ , Z , and A are the density, atomic number, and weight of the material; and β is the muon's velocity factor. The simple model used in Eqs. (4a–c) has been shown to agree with simulations for transverse cooling [17,24].

The equations of motion, Eq. (1) with Eqs. (3) and (4), govern beam evolution in a cooling channel. Such stochastic differential equations are normally treated with a Fokker–Planck equation to solve for average phase-space distribution. For linear dynamics, moment equations provide a simpler alternative by solving the quadratic moments of phase-space distribution, i.e., $\Sigma = \langle XX^T \rangle$ where the average is over the phase space as well as stochastic processes. The moment equations corresponding to Eq. (1) read

$$\frac{d\Sigma}{ds} = (JH + A_D)\Sigma + \Sigma(JH + A_D)^T + B. \quad (8)$$

Here the damping matrix

$$A_D = \frac{A - JA^T J}{2} = -\frac{1}{2} \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 2\eta\hat{b}_s & 0 & -\partial_x\eta & 0 \\ 0 & 0 & \eta & 0 & 0 & 0 \\ -2\eta\hat{b}_s & 0 & 0 & \eta & -\partial_y\eta & 0 \\ 0 & 0 & 0 & 0 & \partial_\delta\eta & 0 \\ \partial_x\eta & 0 & \partial_y\eta & 0 & 0 & \partial_\delta\eta \end{pmatrix} \quad (9)$$

is the dissipative part of A . The Hamiltonian part $(A + JA^T J)/2$ is dropped assuming that it is a negligible perturbation to H . The diffusion matrix

$$B = \text{diag}(0, \chi, 0, \chi, 0, \chi_\delta) \quad (10)$$

arises from the stochastic excitations represented by Ξ .

Before addressing the solution of the moment equations, let us mention an important general theorem on the damping rate of the phase-space volume, known as Robinson's theorem [25] in radiation damping. Without excitation, the 6D emittance $\varepsilon_{6D} \equiv \sqrt{\det \Sigma}$ damps as⁴

$$\varepsilon_{6D}^{-1} \frac{d\varepsilon_{6D}}{ds} = \text{Tr } A_D = -(2\eta + \partial_\delta\eta) \quad (11)$$

which holds for all cooling channels. The damping rates partitioned for individual subspaces can be manipulated by a channel design, but the overall damping is limited by this theorem.

The moment equations represent a coupled evolution of the 21 independent moments in the symmetric 6×6 matrix Σ . In general they are too complicated for analytical treatment. However, the system becomes greatly simplified if material interactions are small perturbations and consideration is limited to the behavior near equilibrium. Our treatment here follows Ref. [11]. Since the interactions with material are weak, the moment equations can be solved perturbatively. First we solve the Hamiltonian evolution

$$\frac{d\Sigma_H}{ds} = (JH)\Sigma_H + \Sigma_H(JH)^T. \quad (12)$$

⁴Proof is straightforward using $d|\Sigma|/ds = \text{Tr}(|\Sigma|\Sigma^{-T} d\Sigma/ds)$ and the moment equations.

The solutions of this homogeneous equation form a linear space, on which an invariant scalar product can be defined as⁵

$$(R, S) = \frac{1}{2} \text{Tr}(RS^+) \quad (13)$$

where $S^+ \equiv -JS^T J$ is the symplectic conjugate of S . With this scalar product, we can define an orthogonal basis $\{\sigma_\alpha\}$ for the solution space, where σ_α satisfies Eq. (12) with periodic boundary conditions and orthogonality $(\sigma_\alpha, \sigma_\beta) = \delta_{\alpha\beta}$, and express any stationary solution as a linear combination of them, i.e.,

$$\Sigma_H = \sum_\alpha \varepsilon_\alpha \sigma_\alpha \quad \text{and} \quad \varepsilon_\alpha = (\Sigma_H, \sigma_\alpha). \quad (14)$$

The periodic solution Σ_H represents the equilibrium Gaussian distribution in the Hamiltonian channel. If a beam initial distribution matches Σ_H , it stays stationary, otherwise it evolves toward Σ_H through filamentation. Our cooling consideration assumes a stationary initial distribution. Note that the σ_α depend on the lattice functions that are determined by the Hamiltonian of a channel. On the other hand ε_α 's, called generalized emittances, are beam properties determined by the beam distribution. Usually there are three emittances in 6D phase space, but there may be a few more if there are degeneracies. Like the 6D emittance ε_{6D} , these subspace emittances are conserved under Hamiltonian flow due to the invariant property of the scalar product. The relation between ε_{6D} and ε_α is nontrivial and depends on the phase-space structure of the Hamiltonian flow. Furthermore, it is easy to see that (XX^T, σ_α) is a quadratic invariant of single-particle motion, whose average leads to emittance. Therefore, finding all linearly independent quadratic invariants effectively

⁵This remarkable scalar product was introduced by Ruggiero et al. [11]. It resembles the usual scalar product of vectors via $\text{Tr}(AB^T) = \sum_{\langle ij \rangle} A_{ij} B_{ij}$. The use of the symplectic conjugate makes it invariant for any symplectic transformation $\tilde{X} = MX$, because the moments transform as $\tilde{\Sigma} = M\Sigma M^T$ and the scalar product transforms as $(\tilde{A}, \tilde{B}) = -\frac{1}{2} \text{Tr}(MAM^T JMBM^T J) = -\frac{1}{2} \text{Tr}(AJBJ) = (A, B)$, where $M^T J M = J$ since M is symplectic. Also note the close connection of this scalar product with many other kinematic moment invariants found around the same time (for example, F. Neri, G. Rangarajan, Phys. Rev. Lett. 64(10) (1990) 1073 and A. Dragt, in Ref. [26], p. 81).

amounts to solving the moment equation (12), and vice versa.

After solving the Hamiltonian evolution, we solve the evolution of the original system, Eq. (8), assuming the same beam envelope but with evolving emittances, i.e., $\Sigma = \sum_{\alpha} \varepsilon_{\alpha}(s) \sigma_{\alpha}$. It can be shown that the errors are next-order perturbations. This leads to the emittance evolution equation

$$\frac{d\varepsilon_k}{ds} = \sum_{\alpha} 2(A_D \sigma_{\alpha}, \sigma_k) \varepsilon_{\alpha} + (B, \sigma_k) \quad (15)$$

where we have used the fact $(A_D \sigma_{\alpha}, \sigma_k) = (\sigma_{\alpha} A_D^T, \sigma_k)$. The two r.h.s. terms describe the damping and excitation, respectively. The balance of these two terms determines the equilibrium emittances, while the coefficients of the damping term determine the damping constants. Note that although Eq. (15) describes only the evolution of a beam with matched initial distribution, the equilibrium emittances depend on neither the initial conditions nor the routes evolving toward the equilibrium. The emittance evolution equation is a triumph of the beam-envelope theory. Comparing to the moment equations, Eq. (15) is a much smaller set of first-order differential equations, yet it determines beam evolution during the cooling process. Furthermore, the emittances evolve much more slowly than the moments and do not change much in a period, thus it is usually a good approximation to average Eq. (15) over a period, which makes it easy to solve. Finally, thanks to the scalar product, Eq. (15) is invariant under canonical transformations.

3. Canonical transformations and lattice functions

To solve for the matrices σ_{α} , we reduce the Hamiltonian Eq. (3) to a simple decoupled form via successive canonical transformations. From the decoupled Hamiltonian, the quadratic invariants and lattice functions become obvious.

First, we remove the angular momentum term by rotating to the Larmor frame that rotates around the longitudinal axis at half of the cyclotron frequency. Using the tilde \sim over a symbol to indicate that it is in the Larmor frame,

the transformation reads

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = T_R \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{y} \\ \tilde{p}_y \end{pmatrix} \quad (16)$$

$$T_R = \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

where $\theta(s) = \int_0^s \hat{b}_s(\bar{s}) d\bar{s}$ is the rotating angle of the Larmor frame. Here we omitted the identity longitudinal transformation in T_R . The generating function reads

$$F_2 = x[\tilde{p}_x \cos \theta + \tilde{p}_y \sin \theta] + y[\tilde{p}_y \cos \theta - \tilde{p}_x \sin \theta]. \quad (17)$$

In the Larmor frame, the Hamiltonian becomes

$$\begin{aligned} \tilde{H} = & \frac{1}{2}(\tilde{p}_x^2 + \tilde{p}_y^2) + \frac{1}{2}(K + K_a \cos 2\theta)\tilde{x}^2 \\ & + \frac{1}{2}(K - K_a \cos 2\theta)\tilde{y}^2 \\ & + K_a \sin 2\theta \tilde{x}\tilde{y} - \frac{\tilde{x}\delta \cos \theta}{\rho} - \frac{\tilde{y}\delta \sin \theta}{\rho} \\ & + \frac{1}{2} \left[\frac{1}{\gamma_0^2} \delta^2 + V z^2 \right] \end{aligned} \quad (18)$$

where the symmetric focusing strength $K = \hat{b}_s(s)^2 + 1/2\rho(s)^2$ and the asymmetric focusing strength $K_a = \hat{b}_1(s) + 1/2\rho(s)^2$.

Second, we remove the coupling between the transverse and longitudinal motions by introducing the dispersions \tilde{D}_x , \tilde{D}_y and a corresponding canonical transformation

$$\begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{y} \\ \tilde{p}_y \\ z \\ \delta \end{pmatrix} = T_D \begin{pmatrix} \tilde{x}_{\beta} \\ \tilde{p}_{x\beta} \\ \tilde{y}_{\beta} \\ \tilde{p}_{y\beta} \\ \hat{z} \\ \hat{\delta} \end{pmatrix}$$

$$T_D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \tilde{D}_x \\ 0 & 1 & 0 & 0 & 0 & \tilde{D}'_x \\ 0 & 0 & 1 & 0 & 0 & \tilde{D}_y \\ 0 & 0 & 0 & 1 & 0 & \tilde{D}'_y \\ -\tilde{D}'_x & \tilde{D}_x & -\tilde{D}'_y & \tilde{D}_y & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (19)$$

which can be generated by the generating function

$$F = (\tilde{x} - \tilde{D}_x \delta) \tilde{p}_{x\beta} + \tilde{D}'_x \tilde{x} \delta + (\tilde{y} - \tilde{D}_y \delta) \tilde{p}_{y\beta} + \tilde{D}'_y \tilde{y} \delta - \hat{z} \delta - \frac{1}{2} (\tilde{D}_x \tilde{D}'_x + \tilde{D}_y \tilde{D}'_y) \delta^2. \quad (20)$$

The transformed Hamiltonian \tilde{H}_β is complicated. However, it can be dramatically simplified and decoupled by requiring (i) the cavity regions are dispersion free and (ii) the dispersion functions satisfy the differential equations⁶

$$\begin{aligned} \tilde{D}''_x + (K + K_a \cos 2\theta) \tilde{D}_x + K_a \sin 2\theta \tilde{D}_y \\ = \frac{\cos \theta}{\rho} \end{aligned} \quad (21a)$$

$$\begin{aligned} \tilde{D}''_y + (K - K_a \cos 2\theta) \tilde{D}_y + K_a \sin 2\theta \tilde{D}_x \\ = \frac{\sin \theta}{\rho}. \end{aligned} \quad (21b)$$

Under these two conditions, the Hamiltonian reduces to

$$\begin{aligned} \tilde{H}_\beta = \frac{1}{2} (\tilde{p}_{x\beta}^2 + \tilde{p}_{y\beta}^2) + \frac{1}{2} (K + K_a \cos 2\theta) \tilde{x}_\beta^2 \\ + \frac{1}{2} (K - K_a \cos 2\theta) \tilde{y}_\beta^2 \\ + K_a \sin 2\theta \tilde{x}_\beta \tilde{y}_\beta + \frac{1}{2} (I \delta^2 + V \hat{z}^2). \end{aligned} \quad (22)$$

Here

$$I(s) = \frac{1}{\gamma_0^2} - \frac{\tilde{D}_x \cos[\theta(s)]}{\rho(s)} - \frac{\tilde{D}_y \sin[\theta(s)]}{\rho(s)}$$

is the negative of the usual phase-slip factor.

Now we still have the $\tilde{x}_\beta \tilde{y}_\beta$ coupling term to remove. Although this is possible in principle with a procedure such as the one in Ref. [27], we will not pursue it here. Instead we discuss two important channels that do not have this coupling

term: quadrupole focusing channels where $\theta = 0$ and solenoidal focusing channels with symmetric focusing where $K_a = 0$. Beside the practical usefulness of these channels, their analytical solutions illustrate important features of ionization cooling and provide formulae for estimating cooling channel performance.

Without the $\tilde{x}_\beta \tilde{y}_\beta$ coupling term, \tilde{H}_β is fully decoupled and the lattice functions for each degree of freedom can be introduced analogous to the well-known Courant–Snyder theory [28]. Let us introduce lattice functions (β_T , α_T , γ_T) for the transverse degrees of freedom, which satisfy the familiar equations

$$\begin{aligned} \beta'_T &= -2\alpha_T, \quad \alpha'_T = K_T(s)\beta_T - \gamma_T, \\ \gamma_T &= \frac{1 + \alpha_T^2}{\beta_T} \end{aligned} \quad (23)$$

where $K_T = K \pm K_a$ for the horizontal or vertical plane in a quadrupole channel and $K_T = K$ for a solenoidal channel. The longitudinal lattice functions (β_L , α_L , γ_L) satisfy similar equations

$$\begin{aligned} \beta'_L &= -2I(s)\alpha_L, \quad \alpha'_L = V(s)\beta_L - I(s)\gamma_L, \\ \gamma_L &= \frac{1 + \alpha_L^2}{\beta_L}. \end{aligned} \quad (24)$$

With periodic boundary conditions, these lattice functions describe the transverse and longitudinal machine ellipses that characterize the betatron and synchrotron oscillations.

The Courant–Snyder invariant for each decoupled subspace (q, p) has the well-known form

$$I = \gamma q^2 + 2\alpha qp + \beta p^2. \quad (25)$$

The corresponding σ -matrix has the generic form

$$\hat{\sigma} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \quad (26)$$

which is the basic building block for constructing the orthogonal set $\{\sigma_\alpha\}$ for 6D phase space. To specify the subspaces, subscripts will be given to $\hat{\sigma}$ and the corresponding lattice functions.

4. Quadrupole cooling channel

In quadrupole channels there is no longitudinal field, the Hamiltonian in Eq. (22)

⁶This is a generalization of the well-known dispersion equation for a quadrupole channel where $\theta = 0$.

reduces to

$$H_\beta = \frac{1}{2}(p_{x\beta}^2 + p_{y\beta}^2) + \frac{1}{2}(K + K_a)x_\beta^2 + \frac{1}{2}(K - K_a)y_\beta^2 + \frac{1}{2}(I\delta^2 + V\dot{z}^2). \quad (27)$$

Usually there is no degeneracy among the three decoupled degrees of freedom. Thus the Hamiltonian H_β has three independent quadratic invariants, two transverse and one longitudinal Courant–Snyder invariants. The three orthogonal σ -matrices in the decoupled frame are

$$\sigma_{x\beta} = \begin{pmatrix} \hat{\sigma}_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{y\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{\sigma}_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{\dot{z}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{\sigma}_z \end{pmatrix}. \quad (28)$$

To compute the coefficients of the emittance evolution equation, Eq. (15), the damping and diffusion matrices and the σ_α matrices need to be transformed to the same frame and then the scalar products computed. The results read

$$\varepsilon'_x = -[\eta - (\partial_x \eta)D_x]\varepsilon_x + \frac{1}{2}\beta_x \chi + \frac{1}{2}\mathcal{H}_x \chi_\delta \quad (29a)$$

$$\varepsilon'_y = -\eta\varepsilon_y + \frac{1}{2}\beta_y \chi \quad (29b)$$

$$\varepsilon'_z = -[\partial_\delta \eta + (\partial_x \eta)D_x]\varepsilon_z + \frac{1}{2}\beta_z \chi_\delta + \frac{1}{2}\gamma_z D_x^2 \chi \quad (29c)$$

where $\mathcal{H}_x \equiv \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D_x'^2$. The 6D emittance $\varepsilon_{6D} = \varepsilon_x \varepsilon_y \varepsilon_z$.

Eq. (29) governs emittance evolution in a quadrupole cooling channel. The vertical equation is the well-known equation [26] for transverse cooling. Since the $\partial_\delta \eta$ term usually provides little longitudinal cooling, the term $(\partial_x \eta)D_x$ is used to partition part of the transverse cooling power to the longitudinal plane, a scheme known as emittance exchange. The two indispensable ingredients for emittance exchange, dispersion D_x and wedged absorber represented by $\partial_x \eta$, show up here in a single product. The $\beta_{x,y}\chi/2$ terms in the transverse equations are the well-known heating terms due to multiple scattering. The $\beta_z \chi_\delta$ term in the longitudinal emittance is analogous to the $\beta_x \chi$

term in the transverse emittance. The last terms in the horizontal and longitudinal equations are extra heating terms first revealed through our systematic treatment [19], which need to be controlled in cooling channel designs. Ref. [29] examined the new transverse heating term due to straggling in the context of current cooling channel designs.

The three emittances in Eq. (29) evolve independently according to first-order inhomogeneous differential equations that can be solved analytically. We refer the reader to Ref. [19] for the details and only list here the equilibrium emittances

$$\varepsilon_x^\infty = \frac{\mathcal{W}_1 + \mathcal{W}_3}{\zeta_1 - \zeta_3}, \quad \varepsilon_y^\infty = \frac{\mathcal{W}_1}{\zeta_1}, \quad \varepsilon_z^\infty = \frac{\mathcal{W}_2 + \mathcal{W}_4}{\zeta_3 + \zeta_4} \quad (30)$$

and the cooling length l at which distance the emittance is damped by e -fold

$$l_x = \frac{\lambda}{\zeta_1 - \zeta_3}, \quad l_y = \frac{\lambda}{\zeta_1}, \quad l_z = \frac{\lambda}{\zeta_3 + \zeta_4}. \quad (31)$$

Here λ is the cooling channel period, ζ 's and \mathcal{W} 's are given by

$$\zeta_1 = \int_0^\lambda ds \eta, \quad \zeta_3 = \int_0^\lambda ds \partial_x \eta D_x, \quad \zeta_4 = \int_0^\lambda ds \partial_\delta \eta \quad (32)$$

and

$$\mathcal{W}_1 \simeq \frac{1}{2} \int_0^\lambda ds \beta_x \chi, \quad \mathcal{W}_2 \simeq \frac{1}{2} \int_0^\lambda ds \beta_z \chi_\delta, \quad \mathcal{W}_3 \simeq \frac{1}{2} \int_0^\lambda ds \mathcal{H}_x \chi_\delta, \quad \mathcal{W}_4 \simeq \frac{1}{2} \int_0^\lambda ds \gamma_z D_x^2 \chi. \quad (33)$$

These integrals are analogs of the well-known radiation integrals of radiation damping in electron storage rings, and could be referred to as ionization integrals, scattering integrals, and straggling integrals.

5. Solenoidal cooling channel

Solenoid channels are the primary candidates for transverse cooling channels [1–4]. To achieve emittance exchange, a dipole field is added to generate dispersion. Since the main solenoid field

continuously rotates the beam and tends to make the beam rotationally symmetric, it is advantageous to have symmetric focusing by setting $K_a = 0$, which can be achieved with gradient dipoles whose quadrupole components are tied to the bending radius as $\hat{b}_1(s) = -1/2\rho(s)^2$. With symmetric focusing, the Hamiltonian in Eq. (22) reduces to

$$\tilde{H}_\beta = \frac{1}{2}(\tilde{p}_{x\beta}^2 + \tilde{p}_{y\beta}^2) + \frac{1}{2}K(\tilde{x}_\beta^2 + \tilde{y}_\beta^2) + \frac{1}{2}(I\delta^2 + V\hat{z}^2). \quad (34)$$

Although the three degrees of freedom are fully decoupled, there are obvious degeneracies in the transverse degrees of freedom, which result in two more invariants other than the Courant–Snyder invariants for each degrees of freedom. One is the canonical angular momentum L_z due to the rotation symmetry of the system. The other more subtle invariant I_{xy} can be found from the Poisson bracket of the canonical angular momentum and a transverse Courant–Snyder invariant. Explicitly, the five invariants read

$$I_x = \gamma_T \tilde{x}_\beta^2 + 2\alpha_T \tilde{x}_\beta \tilde{p}_{x\beta} + \beta_T \tilde{p}_{x\beta}^2 \quad (35a)$$

$$I_y = \gamma_T \tilde{y}_\beta^2 + 2\alpha_T \tilde{y}_\beta \tilde{p}_{y\beta} + \beta_T \tilde{p}_{y\beta}^2 \quad (35b)$$

$$I_z = \gamma_L \hat{z}^2 + 2\alpha_L \hat{z} \delta + \beta_L \delta^2 \quad (35c)$$

$$I_{xy} = \gamma_T \tilde{x}_\beta \tilde{y}_\beta + 2\alpha_T \frac{\tilde{x}_\beta \tilde{p}_{y\beta} + \tilde{y}_\beta \tilde{p}_{x\beta}}{2} + \beta_T \tilde{p}_{x\beta} \tilde{p}_{y\beta} \quad (35d)$$

$$L_z = \tilde{x}_\beta \tilde{p}_{y\beta} - \tilde{y}_\beta \tilde{p}_{x\beta}. \quad (35e)$$

The corresponding σ_α matrices are those in Eq. (28) with $\hat{\sigma}_x = \hat{\sigma}_y = \hat{\sigma}_T$ and the following two that are due to the new invariants

$$\sigma_{xy} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \hat{\sigma}_T & 0 \\ \hat{\sigma}_T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & J_2 & 0 \\ -J_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (36)$$

where J_2 is the 2D symplectic identity matrix. The $\sqrt{2}$ is a normalization factor such that $(\sigma_L, \sigma_L) = (\sigma_{xy}, \sigma_{xy}) = 1$. The explicit relationships among the

σ_α matrices and the invariants in Eq. (35) are $(XX^T, \sigma_x) = I_x/2$, $(XX^T, \sigma_y) = I_y/2$, $(XX^T, \sigma_z) = I_z/2$, $(XX^T, \sigma_{xy}) = I_{xy}/\sqrt{2}$, and $(XX^T, \sigma_L) = L_z/\sqrt{2}$. For convenience we remove the $\sqrt{2}$ by defining the emittances as

$$\varepsilon_\alpha \equiv \frac{1}{2} \langle I_\alpha \rangle \quad (37)$$

the same as the familiar relation in uncoupled motion, instead of $(\langle XX^T \rangle, \sigma_\alpha)$ in Eq. (14). This definition scales ε_{xy} and ε_L as defined in Eq. (14) by a factor of $1/\sqrt{2}$. To be clear, all the subspace emittances used below are defined by Eq. (37).

Transforming the σ_α matrices and the damping and diffusion matrices to the Larmor frame, we compute the scalar products in Eq. (15) and obtain the emittance evolution equation

$$\frac{d\varepsilon_i}{ds} = \sum_{j=1}^5 A_{ij} \varepsilon_j + \chi_i. \quad (38)$$

The damping constant matrix A reads

$$A = \begin{pmatrix} -\eta_x & 0 & c_1 & c_2 & 0 \\ 0 & -\eta_y & c_1 & c_2 & 0 \\ \frac{c_1}{2} & \frac{c_1}{2} & -\eta_{xy} & 0 & 0 \\ \frac{c_2}{2} & \frac{c_2}{2} & 0 & -\eta_L & 0 \\ 0 & 0 & 0 & 0 & -\eta_z \end{pmatrix} \quad (39)$$

where

$$\eta_x = \eta - (\partial_{\tilde{x}} \eta) \tilde{D}_x \quad (40a)$$

$$\eta_y = \eta - (\partial_{\tilde{y}} \eta) \tilde{D}_y \quad (40b)$$

$$\eta_{xy} = \eta_L = \eta - \frac{1}{2} [(\partial_{\tilde{x}} \eta) \tilde{D}_x + (\partial_{\tilde{y}} \eta) \tilde{D}_y] \quad (40c)$$

$$\eta_z = \partial_\delta \eta + (\partial_{\tilde{x}} \eta) \tilde{D}_x + (\partial_{\tilde{y}} \eta) \tilde{D}_y \quad (40d)$$

$$c_1 = \frac{1}{2} [(\partial_{\tilde{x}} \eta) \tilde{D}_y + (\partial_{\tilde{y}} \eta) \tilde{D}_x] \quad (40e)$$

$$c_2 = \hat{b}_s \beta_T \eta + \frac{1}{2} [(\partial_{\tilde{x}} \eta)(\alpha_T \tilde{D}_y + \beta_T \tilde{D}'_y) - (\partial_{\tilde{y}} \eta)(\alpha_T \tilde{D}_x + \beta_T \tilde{D}'_x)] \quad (40f)$$

and the excitations read

$$\chi_x = \frac{1}{2} \beta_T \chi + \frac{1}{2} \mathcal{H}_x \chi_\delta \quad (41a)$$

$$\chi_y = \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_y\chi_\delta \quad (41b)$$

$$\chi_{xy} = \frac{1}{2}\mathcal{H}_{xy}\chi_\delta \quad (41c)$$

$$\chi_L = \frac{1}{2}\mathcal{H}_L\chi_\delta \quad (41d)$$

$$\chi_z = \frac{1}{2}\beta_L\chi_\delta + \frac{1}{2}\gamma_L(\tilde{D}_x^2 + \tilde{D}_y^2)\chi. \quad (41e)$$

The \mathcal{H} -functions (\mathcal{H}_x , \mathcal{H}_y , \mathcal{H}_{xy} , and \mathcal{H}_L) are defined by replacing the phase-space variables in the invariants of Eq. (35) with the dispersion functions and their derivatives. For example, as in radiation damping theory, $\mathcal{H}_x = \gamma_T\tilde{D}_x^2 + 2\alpha_T\tilde{D}_x\tilde{D}'_x + \beta_T\tilde{D}_x'^2$.

To emphasize the rotation symmetry and simplify the results, we use the symmetric and asymmetric invariants $I_{s,a} = (I_x \pm I_y)/2$ and emittances $\varepsilon_{s,a} = (\varepsilon_x \pm \varepsilon_y)/2$. Using these definitions, the equilibrium Gaussian distribution reads

$$\rho(X) = \frac{1}{(2\pi)^3\varepsilon_{6D}} e^{-\frac{\varepsilon_s I_s - \varepsilon_a I_a - \varepsilon_{xy} I_{xy} - \varepsilon_L I_L}{\varepsilon_s^2 - \varepsilon_a^2 - \varepsilon_{xy}^2 - \varepsilon_L^2} - \frac{I_z}{2\varepsilon_z}} \quad (42)$$

where the 6D emittance relates to the subspace emittances as

$$\varepsilon_{6D} = (\varepsilon_s^2 - \varepsilon_a^2 - \varepsilon_{xy}^2 - \varepsilon_L^2)\varepsilon_z. \quad (43)$$

The emittance evolution equations become

$$\begin{aligned} \varepsilon'_s = & -(\eta - ec_-)\varepsilon_s + ec_+\varepsilon_a + es_+\varepsilon_{xy} \\ & + b\varepsilon_L + \chi_s \end{aligned} \quad (44a)$$

$$\varepsilon'_a = -(\eta - ec_-)\varepsilon_a + ec_+\varepsilon_s + \chi_a \quad (44b)$$

$$\varepsilon'_{xy} = -(\eta - ec_-)\varepsilon_{xy} + es_+\varepsilon_s + \chi_{xy} \quad (44c)$$

$$\varepsilon'_L = -(\eta - ec_-)\varepsilon_L + b\varepsilon_s + \chi_L \quad (44d)$$

$$\varepsilon'_z = -(\partial_\delta\eta + 2ec_-)\varepsilon_z + \chi_z \quad (44e)$$

where $e = |\vec{D}| \cdot |\vec{\partial}\eta|/2$ is half of the maximum exchange rate through dispersions and wedges, $c_\pm = \cos(\theta_D \pm \theta_W)$ and $s_\pm = \sin(\theta_D \pm \theta_W)$ with θ_D and θ_W being the orientations of the dispersion vector and the wedges, and $b = \eta\hat{b}_s\beta_T + \alpha_T\varepsilon_s - \beta_T\varepsilon'_s$ with $\varepsilon' = |\vec{D}'| \cdot |\vec{\partial}\eta|/2$ and $s'_- = \sin(\theta_{D'} - \theta_W)$. The excitation terms for the symmetric and

asymmetric emittances are

$$\chi_s = \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_s\chi_\delta \quad (45a)$$

$$\chi_a = \frac{1}{2}\mathcal{H}_a\chi_\delta. \quad (45b)$$

Note that the energy straggling heats all subspaces while the multiple scattering heats only the symmetric transverse emittance and the longitudinal emittance.

The emittance evolution equation (44), shows the same feature of emittance exchange between the transverse and longitudinal degrees of freedom as in Eq. (29). Note that solenoidal channels have the advantage that the reduction of the transverse cooling rate is evenly shared in both transverse planes and thus is only half of the value in a quadrupole channel with dispersion in one plane. The longitudinal emittance evolution is still independent of the transverse evolution, but the evolution of transverse emittances is coupled due to the degeneracy. As mentioned before, it is usually a good approximation to average the emittance evolution equation over a period. After averaging, the equilibrium emittances can be solved by straightforward matrix inversion as

$$\varepsilon_s^{\text{eq.}} \simeq \frac{\overline{\eta - ec_-}\tilde{\chi}_s + \overline{ec_+}\tilde{\chi}_a + \overline{es_+}\tilde{\chi}_{xy} + \overline{b}\tilde{\chi}_L}{\overline{\eta - ec_-}^2 - \overline{ec_+}^2 - \overline{es_+}^2 - \overline{b}^2} \quad (46a)$$

$$\varepsilon_a^{\text{eq.}} \simeq (\overline{ec_+}\varepsilon_s^{\text{eq.}} + \tilde{\chi}_a)/\overline{\eta - ec_-} \quad (46b)$$

$$\varepsilon_{xy}^{\text{eq.}} \simeq (\overline{es_+}\varepsilon_s^{\text{eq.}} + \tilde{\chi}_{xy})/\overline{\eta - ec_-} \quad (46c)$$

$$\varepsilon_L^{\text{eq.}} \simeq (\overline{b}\varepsilon_s^{\text{eq.}} + \tilde{\chi}_L)/\overline{\eta - ec_-} \quad (46d)$$

$$\varepsilon_z^{\text{eq.}} \simeq \tilde{\chi}_z/(\overline{\partial_\delta\eta} + 2\overline{ec_-}). \quad (46e)$$

Here the overline indicates averaging over a period. The longitudinal cooling length is $\overline{\partial_\delta\eta} + 2\overline{ec_-}^{-1}$. Due to coupling, the cooling length for each transverse emittance is not unique but has three different values: $\overline{\eta - ec_-}^{-1}$ and $(\overline{\eta - ec_-} \pm \sqrt{\overline{ec_+}^2 + \overline{es_+}^2 + \overline{b}^2})^{-1}$. Clearly one may introduce various integrals to characterize the cooling process in a solenoidal channel. In cooling channel designs, one should minimize $\overline{ec_+}$, $\overline{es_+}$, and \overline{b} as well as the excitations. See more discussion on transverse cooling in Ref. [17] and on 6D cooling in Ref. [20].

The emittance evolution equation, Eq. (44), is derived for solenoidal channels with symmetric focusing and no dispersion in the rf field. However, it may still provide a useful estimate for the proposed cooling channels that seem not satisfy these conditions, because the focusing from solenoids is much stronger than the weak focusing from bending dipoles, and the muons' longitudinal position variation due to dispersion ($\vec{D} \cdot \vec{P} - \vec{D}' \cdot \vec{X}$) is much smaller than the rf wavelength.

6. Conclusion

We presented a linear beam-envelope theory of ionization cooling. A small set of emittance evolution equations that governs the cooling process is derived. Equilibrium emittances and damping constants are obtained. The systematic treatment of the cooling and heating mechanisms revealed new heating sources. The analytical results can be used to evaluate cooling channel designs the same way that radiation integrals are used in storage ring designs. Further improvements may take into account the longitudinal nonlinear effects due to the fact that, in all proposed cooling channels, muons fill the whole rf bucket instead of just the linear core area.

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